

New approach to Minkowski fractional inequalities using generalized k-fractional integral operator

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Abstract

In this paper, we obtain new results related to Minkowski fractional integral inequality using generalized k-fractional integral operator which is in terms of the Gauss hypergeometric function.

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1 Introduction

In the last decades many researchers have worked on fractional integral inequalities using Riemann-Liouville, generalized Riemann-Liouville, Hadamard and Saigo, see [2, 3, 4, 7, 8, 9, 10]. W. Yang [23] proved the Chebyshev and Grüss-type integral inequalities for Saigo fractional integral operator. S. Mubeen and S. Iqbal [14] has proved the Grüss-type integral inequalities generalized k-fractional integral. In [1, 6, 13, 24] authors have studied some fractional integral inequalities using generalized k-fractional integral operator (in terms of the Gauss hypergeometric function). Recently many researchers have shown development of fractional integral inequalities associated with hypergeometric functions, see [11, 13, 15, 17, 18, 19, 21, 22, 24]. Also, in [3, 7] authors established reverse Minkowski fractional integral inequality using Hadamard and Riemann-Liouville integral operator respectively.

In literature few results have been obtained on some fractional integral inequalities using Saigo fractional integral operator, see [5, 12, 15, 16, 24]. Motivated from [2, 6, 7, 13], our purpose in this paper is to establish some new results using generalized k-fractional integral in terms of Gauss hypergeometric function. The paper has been organized as follows, in section 2, we define basic definitions and proposition related to generalized k-fractional integral.

In section 3, we give the results about reverse Minkowski fractional integral inequality using fractional generalized k-fractional integral, In section 4, we give some other inequalities using fractional generalized k-fractional integral.

2 Preliminaries

In this section, we give some necessary definitions which will be used latter.

Definition 2.1 [13, 24] The function $f(x)$, for all $x > 0$ is said to be in the $L_{p,k}[0, \infty)$, if

$$L_{p,k}[0, \infty) = \left\{ f : \|f\|_{L_{p,k}[0, \infty)} = \left(\int_0^\infty |f(x)|^p x^k dx \right)^{\frac{1}{p}} < \infty \quad 1 \leq p < \infty \quad k \geq 0 \right\}, \quad (2.1)$$

Definition 2.2 [13, 20, 24] Let $f \in L_{1,k}[0, \infty)$,. The generalized Riemann-Liouville fractional integral $I^{\alpha,k}f(x)$ of order $\alpha, k \geq 0$ is defined by

$$I^{\alpha,k}f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x^{k+1} - t^{k+1})^{\alpha-1} t^k f(t) dt. \quad (2.2)$$

Definition 2.3 [13, 24] Let $k \geq 0, \alpha > 0, \mu > -1$ and $\beta, \eta \in R$. The generalized k-fractional integral $I_{x,k}^{\alpha,\beta,\eta,\mu}$ (in terms of the Gauss hypergeometric function) of order α for real-valued continuous function $f(t)$ is defined by

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[f(x)] = \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f(\tau) d\tau. \quad (2.3)$$

where, the function ${}_2F_1(-)$ in the right-hand side of (2.3) is the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (2.4)$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0 = 1.$$

Consider the function

$$\begin{aligned}
F(x, \tau) &= \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \tau^{(k+1)\mu} \\
&= (x^{k+1} - \tau^{k+1})^{\alpha-1} \times {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \\
&= \sum_{n=0}^{\infty} \frac{(\alpha + \beta + \mu)_n (-n)_n}{\Gamma(\alpha + n) n!} x^{(k+1)(-\alpha-\beta-2\mu-\eta)} \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1+n} (k+1)^{\mu+\beta+1} \\
&= \frac{\tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} (k+1)^{\mu+\beta+1}}{x^{k+1} (\alpha + \beta + 2\mu) \Gamma(\alpha)} + \\
&\quad \frac{\tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha} (k+1)^{\mu+\beta+1} (\alpha + \beta + \mu) (-n)}{x^{k+1} (\alpha + \beta + 2\mu + 1) \Gamma(\alpha + 1)} + \\
&\quad \frac{\tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha+1} (k+1)^{\mu+\beta+1} (\alpha + \beta + \mu) (\alpha + \beta + \mu + 1) (-n) (-n + 1)}{x^{k+1} (\alpha + \beta + 2\mu + 1) \Gamma(\alpha + 2) 2!} + \dots
\end{aligned} \tag{2.5}$$

It is clear that $F(x, \tau)$ is positive because for all $\tau \in (0, x)$, $(x > 0)$ since each term of the (2.5) is positive.

3 Reverse Minkowski fractional integral inequality

In this section, we establish reverse Minkowski fractional integral inequality using generalized k-fractional integral operator (in terms of the Gauss hypergeometric function).

Theorem 3.1 *Let $p \geq 1$ and let f, g be two positive function on $[0, \infty)$, such that for all $x > 0$, $I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] < \infty$, $I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0, x)$ we have*

$$\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{1}{p}} + \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{1}{p}} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^p(x)] \right]^{\frac{1}{p}}, \tag{3.1}$$

for all $k \geq 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\beta < 1$, $\mu > -1$, $\beta - 1 < \eta < 0$.

Proof: Using the condition $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0, x)$, $x > 0$, we can write

$$(M+1)^p f(\tau) \leq M^p (f+g)^p(\tau). \tag{3.2}$$

Multiplying both side of (3.2) by $F(x, \tau)$, then integrating resulting identity with respect to τ from 0 to x , we get

$$\begin{aligned}
& (M+1)^p \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f^p(\tau) d\tau \\
& \leq M^p \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k (f+g)^p(\tau) d\tau,
\end{aligned} \tag{3.3}$$

which is equivalent to

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \leq \frac{M^p}{(M+1)^p} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^p(x)] \right], \tag{3.4}$$

hence, we can write

$$\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{1}{p}} \leq \frac{M}{(M+1)} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^p(x)] \right]^{\frac{1}{p}}. \tag{3.5}$$

On other hand, using condition $m \leq \frac{f(\tau)}{g(\tau)}$, we obtain

$$(1 + \frac{1}{m})g(\tau) \leq \frac{1}{m}(f(\tau) + g(\tau)), \tag{3.6}$$

therefore,

$$(1 + \frac{1}{m})^p g^p(\tau) \leq (\frac{1}{m})^p (f(\tau) + g(\tau))^p. \tag{3.7}$$

Now, multiplying both side of (3.7) by $F(x, \tau)$, ($\tau \in (0, x)$, $x > 0$), where $G(x, \tau)$ is defined by (2.5). Then integrating resulting identity with respect to τ from 0 to x , we have

$$\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^p(x)] \right]^{\frac{1}{p}} \leq \frac{1}{(m+1)} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^p(x)] \right]^{\frac{1}{p}}. \tag{3.8}$$

The inequalities (3.1) follows on adding the inequalities (3.5) and (3.8).

Our second result is as follows.

Theorem 3.2 *Let $p \geq 1$ and f, g be two positive function on $[0, \infty)$, such that for all $x > 0$, $I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] < \infty$, $I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] < \infty$. If $0 < m \leq$*

$\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0, x)$ then we have

$$\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{2}{p}} + \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{2}{p}} \geq \left(\frac{(M+1)(m+1)}{M} - 2 \right) \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{1}{p}} + \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{1}{p}}. \quad (3.9)$$

for all $k \geq 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\beta < 1$, $\mu > -1$, $\beta - 1 < \eta < 0$.

Proof: Multiplying the inequalities (3.5) and (3.8), we obtain

$$\frac{(M+1)(m+1)}{M} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{1}{p}} \times \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{1}{p}} \leq \left(\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f(x) + g(x))^p] \right]^{\frac{1}{p}} \right)^2. \quad (3.10)$$

Applying Minkowski inequalities to the right hand side of (3.10), we have

$$\left(\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f(x) + g(x))^p] \right]^{\frac{1}{p}} \right)^2 \leq \left(\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{1}{p}} + \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{1}{p}} \right)^2, \quad (3.11)$$

which implies that

$$\begin{aligned} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[(f(x) + g(x))^p] \right]^{\frac{2}{p}} &\leq \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{2}{p}} + \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{2}{p}} \\ &\quad + 2 \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \right]^{\frac{1}{p}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \right]^{\frac{1}{p}}. \end{aligned} \quad (3.12)$$

Hence, from (3.10) and (3.12), we obtain (3.9). Theorem 3.2 is thus proved.

4 Other fractional integral inequalities related to Minkowski inequality

In this section, we establish some new integral inequalities related to Minkowski inequality using generalized k-fractional integral operator (in terms of the Gauss hypergeometric function).

Theorem 4.1 Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and f, g be two positive function on $[0, \infty)$, such that $I_{x,k}^{\alpha,\beta,\eta,\mu}[f(x)] < \infty$, $I_{x,k}^{\alpha,\beta,\eta,\mu}[g(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < \infty$, $\tau \in [0, x]$ we have

$$\left[I_{x,k}^{\alpha,\beta,\eta,\mu}[f(x)] \right]^{\frac{1}{p}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu}[g(x)] \right]^{\frac{1}{q}} \leq \left(\frac{M}{m} \right)^{\frac{1}{pq}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \right], \quad (4.1)$$

for all $k \geq 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\beta < 1$, $\mu > -1$, $\beta - 1 < \eta < 0$.

Proof:- Since $\frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in [0, x]$ $x > 0$, therefore

$$[g(\tau)]^{\frac{1}{p}} \geq M^{\frac{-1}{q}} [f(\tau)]^{\frac{1}{q}}, \quad (4.2)$$

and also,

$$\begin{aligned} [f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} &\geq M^{\frac{-1}{q}} [f(\tau)]^{\frac{1}{q}} [f(\tau)]^{\frac{1}{p}} \\ &\geq M^{\frac{-1}{q}} [f(\tau)]^{\frac{1}{q} + \frac{1}{p}} \\ &\geq M^{\frac{-1}{q}} [f(\tau)]. \end{aligned} \quad (4.3)$$

Multiplying both side of (4.3) by $F(x, \tau)$, ($\tau \in (0, x)$, $x > 0$), where $F(x, \tau)$ is defined by (2.5). Then integrating resulting identity with respect to τ from 0 to x , we have

$$\begin{aligned} &\frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ &{}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f(\tau)^{\frac{1}{p}} g(\tau)^{\frac{1}{q}} d\tau \\ &\leq M^{\frac{-1}{q}} \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ &{}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f(\tau) d\tau, \end{aligned} \quad (4.4)$$

which implies that

$$I_{x,k}^{\alpha,\beta,\eta,\mu} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \leq M^{\frac{-1}{q}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} f(x) \right]. \quad (4.5)$$

Consequently,

$$\left(I_{x,k}^{\alpha,\beta,\eta,\mu} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \right)^{\frac{1}{p}} \leq M^{\frac{-1}{pq}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} f(x) \right]^{\frac{1}{p}}, \quad (4.6)$$

on other hand, since $mg(\tau) \leq f(\tau)$, $\tau \in [0, x]$, $x > 0$, then we have

$$[f(\tau)]^{\frac{1}{p}} \geq m^{\frac{1}{p}} [g(\tau)]^{\frac{1}{p}}, \quad (4.7)$$

multiplying equation (4.7) by $[g(\tau)]^{\frac{1}{q}}$, we have

$$[f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} \geq m^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} [g(\tau)]^{\frac{1}{p}} = m^{\frac{1}{p}} [g(\tau)]. \quad (4.8)$$

Multiplying both side of (4.8) by $F(x, \tau)$, ($\tau \in (0, x)$, $x > 0$), where $F(x, \tau)$ is defined by (2.5). Then integrating resulting identity with respect to τ from

0 to x , we have

$$\begin{aligned}
& \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f(\tau)^{\frac{1}{p}} g(\tau)^{\frac{1}{q}} d\tau \\
& \leq M^{\frac{1}{p}} \frac{(k+1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k g(\tau) d\tau,
\end{aligned} \tag{4.9}$$

which implies that

$$I_{x,k}^{\alpha,\beta,\eta,\mu} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \leq m^{\frac{1}{p}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} g(x) \right]. \tag{4.10}$$

Hence, we can write

$$\left(I_{x,k}^{\alpha,\beta,\eta,\mu} \left[[f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right] \right)^{\frac{1}{q}} \leq m^{\frac{1}{pq}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} f(x) \right]^{\frac{1}{q}}, \tag{4.11}$$

multiplying equation (4.6) and (4.11) we get the result (4.1).

Theorem 4.2 *Let f and g be two positive function on $[0, \infty[$, such that $I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] < \infty$, $I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] < \infty$. $x > 0$, If $0 < m \leq \frac{f(\tau)^p}{g(\tau)^q} \leq M < \infty$, $\tau \in [0, x]$. Then we have*

$$\left[I_{x,k}^{\alpha,\beta,\eta,\mu} f^p(x) \right]^{\frac{1}{p}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} g^q(x) \right]^{\frac{1}{q}} \leq \left(\frac{M}{m} \right)^{\frac{1}{pq}} \left[I_{x,k}^{\alpha,\beta,\eta,\mu} (f(x)g(x)) \right] \text{ hold.}$$

Where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, for all $k \geq 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\beta < 1$, $\mu > -1$, $\beta - 1 < \eta < 0$.

Proof:- Replacing $f(\tau)$ and $g(\tau)$ by $f(\tau)^p$ and $g(\tau)^q$, $\tau \in [0, x]$, $x > 0$ in theorem 4.1, we obtain required inequality.

Now, here we present fractional integral inequality related to Minkowsky inequality as follows

Theorem 4.3 *let f and g be two integrable functions on $[1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and $0 < m < \frac{f(\tau)}{g(\tau)} < M$, $\tau \in (0, x)$, $x > 0$. Then for all $\alpha > 0$, we have*

$$I_{x,k}^{\alpha,\beta,\eta,\mu} \{fg\}(x) \leq \frac{2^{p-1} M^p}{p(M+1)^p} \left(I_{x,k}^{\alpha,\beta,\eta,\mu} [f^p + g^p](x) \right) + \frac{2^{q-1}}{q(m+1)^q} \left(I_{x,k}^{\alpha,\beta,\eta,\mu} [f^q + g^q](x) \right), \tag{4.12}$$

for all $k \geq 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\beta < 1$, $\mu > -1$, $\beta - 1 < \eta < 0$.

Proof:- Since, $\frac{f(\tau)}{g(\tau)} < M, \tau \in (0, x), x > 0$, we have

$$(M + 1)f(\tau) \leq M(f + g)(\tau). \quad (4.13)$$

Taking p^{th} power on both side and multiplying resulting identity by $F(x, \tau)$, we obtain

$$\begin{aligned} & (M + 1)^p \frac{(k + 1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ & {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f^p(\tau) d\tau \\ & \leq M^p \frac{(k + 1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ & {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k (f + g)^p(\tau) d\tau, \end{aligned} \quad (4.14)$$

therefore,

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] \leq \frac{M^p}{(M + 1)^p} I_{x,k}^{\alpha,\beta,\eta,\mu}[(f + g)^p(x)], \quad (4.15)$$

on other hand, $0 < m < \frac{f(\tau)}{g(\tau)}, \tau \in (0, x), x > 0$, we can write

$$(m + 1)g(\tau) \leq (f + g)(\tau), \quad (4.16)$$

therefore,

$$\begin{aligned} & (m + 1)^q \frac{(k + 1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ & {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k g^q(\tau) d\tau \\ & \leq \frac{(k + 1)^{\mu+\beta+1} x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu} (x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\ & {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k (f + g)^q(\tau) d\tau, \end{aligned} \quad (4.17)$$

consequently, we have

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)] \leq \frac{1}{(m + 1)^q} I_{x,k}^{\alpha,\beta,\eta,\mu}[(f + g)^q(x)]. \quad (4.18)$$

Now, using Young inequality

$$[f(\tau)g(\tau)] \leq \frac{f^p(\tau)}{p} + \frac{g^q(\tau)}{q}. \quad (4.19)$$

Multiplying both side of (4.19) by $F(x, \tau)$, which is positive because $\tau \in (0, x)$, $x > 0$, then integrate the resulting identity with respect to τ from 0 to x , we get

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[f(x)g(x)] \leq \frac{1}{p} I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p(x)] + \frac{1}{q} I_{x,k}^{\alpha,\beta,\eta,\mu}[g^q(x)], \quad (4.20)$$

from equation (4.15), (4.18) and (4.20) we get

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[f(x)g(x)] \leq \frac{M^p}{p(M+1)^p} I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^p(x)] + \frac{1}{q(m+1)^q} I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^q(x)], \quad (4.21)$$

now using the inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$, $r > 1$, $a, b \geq 0$, we have

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^p(x)] \leq 2^{p-1} I_{x,k}^{\alpha,\beta,\eta,\mu}[f^p + g^p](x), \quad (4.22)$$

and

$$I_{x,k}^{\alpha,\beta,\eta,\mu}[(f+g)^q(x)] \leq 2^{q-1} I_{x,k}^{\alpha,\beta,\eta,\mu}[f^q + g^q](x). \quad (4.23)$$

Injecting (4.22), (4.23) in (4.21) we get required inequality (4.12). This complete the proof.

Theorem 4.4 *Let f, g be two positive function on $[0, \infty)$, such that f is non-decreasing and g is non-increasing. Then*

$$\begin{aligned} I_{x,k}^{\alpha,\beta,\eta,\mu} f^\gamma(x) g^\delta(x) &\leq (k+1)^{-\mu-\beta} x^{(k+1)(\mu+\beta)} \frac{\Gamma(1-\beta)\Gamma(1+\mu+\eta+1)}{\Gamma(1-\beta+\eta)\Gamma(\mu+1)} \\ &\times I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)] I_{x,k}^{\alpha,\beta,\eta,\mu}[g^\delta(x)], \end{aligned} \quad (4.24)$$

for all $k \geq 0$, $\alpha > \max\{0, -\beta - \mu\}$, $\beta < 1$, $\mu > -1$, $\beta - 1 < \eta < 0$.

Proof:- let $\tau, \rho \in [0, x]$, $x > 0$, for any $\delta > 0$, $\gamma > 0$, we have

$$(f^\gamma(\tau) - f^\gamma(\rho)) (g^\delta(\rho) - g^\delta(\tau)) \geq 0, \quad (4.25)$$

$$f^\gamma(\tau)g^\delta(\rho) - f^\gamma(\tau)g^\delta(\tau) - f^\gamma(\rho)(g^\delta(\rho) + f^\gamma(\rho)g^\delta(\tau)) \geq 0, \quad (4.26)$$

therefore

$$f^\gamma(\tau)g^\delta(\tau) + f^\gamma(\rho)(g^\delta(\rho) \leq f^\gamma(\tau)g^\delta(\rho) + f^\gamma(\rho)g^\delta(\tau). \quad (4.27)$$

Now, multiplying both side of (4.27) by $F(x, \tau)$, ($\tau \in (0, x)$, $x > 0$), where $F(x, \tau)$ is defined by (2.5). Then integrating resulting identity with respect

to τ from 0 to x , we have

$$\begin{aligned}
& \frac{(k+1)^{\mu+\beta+1}x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu}(x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k [f^\gamma(\tau)g^\delta(\tau)] d\tau \\
& + f^\gamma(\rho)g^\delta(\rho) \frac{(k+1)^{\mu+\beta+1}x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu}(x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k [1] d\tau \\
& \leq g^\delta(\rho) \frac{(k+1)^{\mu+\beta+1}x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu}(x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k f^\gamma(\tau) d\tau \\
& + f^\gamma(x) \frac{(k+1)^{\mu+\beta+1}x^{(k+1)(-\alpha-\beta-2\mu)}}{\Gamma(\alpha)} \int_0^x \tau^{(k+1)\mu}(x^{k+1} - \tau^{k+1})^{\alpha-1} \times \\
& {}_2F_1(\alpha + \beta + \mu, -\eta; \alpha; 1 - (\frac{\tau}{x})^{k+1}) \tau^k g^\delta(\tau) d\tau,
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
& I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)g^\delta(x)] + f^\gamma(\rho)(g^\delta(\rho)I_{x,k}^{\alpha,\beta,\eta,\mu}[1] \\
& \leq g^\delta(\rho)I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)] + f^\gamma(\rho)I_{x,k}^{\alpha,\beta,\eta,\mu}[g^\delta(x)].
\end{aligned} \tag{4.29}$$

Again, multiplying both side of (4.29) by $F(x, \rho)$, ($\rho \in (0, x)$, $x > 0$), where $F(x, \rho)$ is defined by (2.5). Then integrating resulting identity with respect to ρ from 0 to x , we have

$$\begin{aligned}
& I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)g^\delta(x)]I_{x,k}^{\alpha,\beta,\eta,\mu}[1] + I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)g^\delta(x)]I_{x,k}^{\alpha,\beta,\eta,\mu}[1] \\
& \leq I_{x,k}^{\alpha,\beta,\eta,\mu}[g^\delta(x)]I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)] + I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)]I_{x,k}^{\alpha,\beta,\eta,\mu}[g^\delta(x)],
\end{aligned}$$

then we can write

$$2I_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)g^\delta(x)] \leq \frac{1}{[I_{x,k}^{\alpha,\beta,\eta,\mu}[1]]^{-1}} 2II_{x,k}^{\alpha,\beta,\eta,\mu}[f^\gamma(x)]I_{x,k}^{\alpha,\beta,\eta,\mu}[g^\delta(x)].$$

This proves the result (4.24).

Competing interests

The authors declare that they have no competing interests.

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